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The τ-invariant and elimination

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ABSTRACT

In this paper we present some results showing the good behavior of the τ-invariant of a Rees algebra with integral closure and elimination (of variables).

1. Introduction

Hironaka's Theorem of embedded desingularization [15] was proven, over fields of characteristic zero, by induction on the dimension of the ambient space. It makes use of the existence of some special smooth hypersurfaces (*hypersurfaces of maximal contact*). The problem of resolution of singularities is reformulated as an equivalent problem in these smooth hypersurfaces, that is, in a smooth scheme in one dimension less. This form of induction holds exclusively over fields of characteristic zero, but fails over fields of positive characteristic (see [12]).

Results on resolution of singularities in positive characteristic in small dimension are due to Abhyankar [1,2]. More recently some strategies to deal with resolution of singularities over arbitrary fields have appeared in works of Kawanoue and Matsuki [18,19], Hironaka [17], Włodarczyk [26], Cossart and Piltant [8,9], Hauser [14], Cutkosky [10], Villamayor [23,25], and Bravo and Villamayor [7].

These last three papers make use of a form of induction in which hypersurfaces of maximal contact are replaced by generic projections on smooth schemes of lower dimension; and restrictions to these

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smooth hypersurfaces are replaced by a different form of elimination of variables. In this paper, we focus on this new form of induction.

When treating problems of embedded resolution of singularities, it is natural to identify ideals with the same integral closure, for example when dealing with embedded principalization, also called Log-resolution of ideals, and also in Hironaka's notion of idealistic exponents.

In this paper embedded ideals and idealistic exponents are reformulated as Rees algebras. The analog of two ideals with the same integral closure (as ideals) will be that of two Rees algebras with the same integral closure (as algebras).

In Theorem 5.2 it is shown that the main invariant considered in most inductive arguments (the Hironaka's τ-invariant) is compatible with integral closure of algebras. A similar result is treated in Kawanoue's work, [18], using an alternative approach. It is proved the compatibility of τ-invariant with integral closure.

The main result of this work, Theorem 6.4, studies the behavior of Hironaka's τ-invariant with the form of elimination mentioned above.

The τ-invariant appears in [15], and it is defined by geometrical conditions: given a hypersurface X and a closed point $x \in X$, the τ-invariant is the codimension of a linear subspace attached to the tangent cone of X at x (see 4.3). For an algebraic point of view, τ is the minimum number of variables needed to express generators of $\mathbb{I}_X$, where $\mathbb{I}_X$ denotes the homogeneous ideal spanned by initial forms of X at x.

Many aspects of the τ-invariant and some of its properties are deeply studied in works of Oda [20,21] or [22]. It also appears in [16] and [18].

The main difference between the results treated here and the study of the τ-invariant developed in the previous papers lies in the alternative way of induction provided by the elimination. In the latest, the τ-invariant is not study in an inductive manner, as it is done here.

The first sections of this paper presents the basic definitions of Rees algebras and its relationship with integral closure and differential operators (for more details see [11] and [24]); Elimination algebras are described in a synthetic manner in Section 3 following Villamayor's ideas; generic projections and formal definitions of tangent cones and τ-invariant are introduced in Section 4.

In Section 5 we prove that two integrally equivalent Rees algebras (i.e. with the same integral closure) have the same τ-invariant. This result, together with a form of local presentation (Proposition 6.3), will be the key to prove the main theorem of this work.

In Section 6 the main result (Theorem 6.4) is addressed: Given a differential Rees algebra so that, locally at a closed point x, $\tau_{\mathcal{G},x} \geqslant 1$, and if we are under conditions to consider its elimination algebra, say $\mathcal{R}_{\mathcal{G}}$, then the τ-invariant drops by one, i.e., $\tau_{\mathcal{R}_{\mathcal{G}}} = \tau_{\mathcal{G}} - 1$.

The importance of this result lies in the inductive arguments that can be considered (analogous to the ones in characteristic zero): There is a well-known notion of resolution of simple Rees algebras by decreasing induction in τ. The main idea is to prove resolution of a simple algebra, using the inductive hypothesis that the resolution can be achieved for algebras with greater τ. For this reason it is important to have a control of the τ-invariant when dealing with elimination (decreasing of the dimension of the ambient space).

In [23, Proposition 5.12] it is proven that given a differential Rees algebra $\mathcal{G}$ so that $\tau_{\mathcal{G},x} > 1$, then its elimination algebra, say $\mathcal{R}_{\mathcal{G}}$, is such that $\tau_{\mathcal{R}_{\mathcal{G}}} \geqslant 1$. Here, Theorem 6.4 provides a more general result, and also generalizes to any arbitrary characteristic $p \geqslant 0$ the well-known property of dropping by one in characteristic zero.

The importance of this result appears in a recent work of Bravo and Villamayor [7], two joint works of the author together with Villamayor [5,6] and the work of the author [4].

2. Rees algebras

2.1. We start introducing our main tool, Rees algebras, and discussing some nice properties which will be useful throughout this paper.

Definition 2.2. A *Rees algebra* over a smooth scheme V is a locally finite generated sub-algebra of $\mathcal{O}_V[W]$, of the form

$$\mathcal{G} = \bigoplus_{n \geqslant 0} I_n W^n,$$

where $\{I_n\}_{n \geqslant 0}$ is a sequence of sheaves of ideals, and $\mathcal{G}$ is locally a finitely generated $\mathcal{O}_V$-algebra so that

1. $I_0 = \mathcal{O}_V$,
2. each I_n is such that $I_s \cdot I_t \subset I_{s+t}$ for $s, t \geqslant 0$.

Remark 2.3. At any affine open subset $U(\subset V)$, there exists a finite set of elements

$$\mathcal{F} = \{f_1 W^{n_1}, \dots, f_s W^{n_s}\},$$

with $n_i \in \mathbb{Z}_{\geqslant 1}$ and $f_i \in \mathcal{O}_V(U)$, so that the restriction to U is of the form

$$\mathcal{G}(U) = \mathcal{O}_V(U)\big[f_1 W^{n_1}, \dots, f_s W^{n_s}\big]\big(\subset \mathcal{O}_V(U)[W]\big).$$

We say that $\mathcal{F} = \{f_1 W^{n_1}, \dots, f_s W^{n_s}\}$ is a *set of generators* of $\mathcal{G}$ locally at U. Any other element of $fW^n \in \mathcal{G}$ can be express by a weighted homogeneous polynomial of degree n and coefficients in $\mathcal{O}_V(U)$, say $F_n(Y_1, \dots, Y_n)$, where each variable Y_j has weight n_j and $f = F_n(f_1, \dots, f_s)$.

Remark 2.4. Rees algebras and Rees rings are closely related. We say that a Rees algebra is a *Rees ring* if in any affine open subset we can choose a set of generators of $\mathcal{G}$, say $\mathcal{F} = \{f_1 W^{n_1}, \dots, f_s W^{n_s}\}$, such that all the weights n_i are $n_i = 1$.

Rees algebras are integral closures of Rees rings: if N is a positive integer divisible by all the weights n_i, then

$$\mathcal{O}_V(U)\big[f_1 W^{n_1}, \dots, f_s W^{n_s}\big] = \bigoplus_{n \geqslant 0} I_n W^n \big(\subset \mathcal{O}_V(U)[W]\big),$$

is integral over the Rees sub-ring $\mathcal{O}_V(U)[I_N W^N](\subset \mathcal{O}_V(U)[W^N])$.

Definition 2.5. Given two Rees algebras $\mathcal{G}_1 = \bigoplus_{n \geqslant 0} I_n^{(1)} W^n$ and $\mathcal{G}_2 = \bigoplus_{n \geqslant 0} I_n^{(2)} W^n$ we define a binary operation between them denoted by $\odot$. In any affine open subset U, if locally $\mathcal{G}_1 = \mathcal{O}_V(U)[f_1 W^{n_1}, \dots, f_r W^{n_r}]$ and $\mathcal{G}_2 = \mathcal{O}_V(U)[g_1 W^{m_1}, \dots, g_s W^{m_s}]$, then

$$\mathcal{G}_1 \odot \mathcal{G}_2 = \mathcal{O}_V(U)\big[f_1 W^{n_1}, \dots, f_r W^{n_r}, g_1 W^{m_1}, \dots, g_s W^{m_s}\big].$$

This algebra $\mathcal{G}_1 \odot \mathcal{G}_2$ is the smallest algebra containing both $\mathcal{G}_1$ and $\mathcal{G}_2$.

Definition 2.6. Fix a Rees algebra $\mathcal{G} = \bigoplus I_n W^n$. A closed set is attached to $\mathcal{G}$, called the *singular locus of* $\mathcal{G}$,

$$\operatorname{Sing}(\mathcal{G}) := \big\{x \in V \bigm| \nu_x(I_n) \geqslant n, \text{ for each } n \geqslant 1\big\},$$

where $\nu_x(I_n)$ denotes the order of the ideal I_n at the local regular ring $\mathcal{O}_{V,x}$.

The following proposition provides an easier form to compute the singular locus in terms of the generators of $\mathcal{G}$.

Proposition 2.7. *Fix a Rees algebra $\mathcal{G}$ over V. Let $U \subset V$ be an affine open set, and $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ be a set of generators as above. Then*

$$\mathrm{Sing}(\mathcal{G}) \cap U = \bigcap_{1 \leqslant i \leqslant s} \left\{ x \in U \;\middle|\; \mathrm{ord}_x(f_i) \geqslant n_i \right\}.$$

Proof. See [24, Proposition 4.4(2)]. □

2.8. We now introduce an equivalence relation between Rees algebras. This notion is closely related with the equivalence relation Hironaka defined in the ambient of couples (J, b), where $J \subset \mathcal{O}_V$ is a sheaf of ideals and b is a positive integer. Here the pairs (J, b) and (J', b') are equivalent, say $(J, b) \sim (J', b')$, if and only if $J^{b'}$ and $(J')^b$ have the same integral closure.

Definition 2.9. Fix two Rees algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ over the smooth scheme V. The algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *integrally equivalent* if $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure. This equivalence is denoted by $\mathcal{G}_1 \sim \mathcal{G}_2$.

Proposition 2.10. *The singular locus is compatible with integral closure: Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two integrally equivalent Rees algebras over V. Then,*

$$\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2).$$

Proof. See [24, Proposition 4.4(1)]. □

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are *integrally* equivalent on V, the same holds for any open restriction, and also for pull-backs by smooth morphisms $W \longrightarrow V$.

2.11. Differential Rees algebras. Here we introduce a particular class of Rees algebras, called the differential Rees algebras, and discuss some significant properties. Some of these properties of differential Rees algebras have a very important meaning in the context of singularities as we will see.

Given a smooth scheme V over a field k, there is a locally free sheaf over V, defined for each non-negative integer s, called the sheaf of differential operators of order s, and denoted by Diff_k^s, this sheaf is so that

1. for $s = 0$, $\mathit{Diff}_k^0 = \mathcal{O}_V$, and
2. for each $s \geqslant 0$ $\mathit{Diff}_k^s \subset \mathit{Diff}_k^{s+1}$.

Given a sheaf of ideals $J \subset \mathcal{O}_V$, we can define an *extension of the sheaf of ideals* with the help of this sheaf of differential operators. This extension of J, denoted by $\mathit{Diff}_k^s(J)$, is so that over any affine open set U,

$$\mathit{Diff}_k^s(J)(U) = \left\{ D(f) \;\middle|\; D \in \mathit{Diff}_k^s(U) \text{ and } f \in J(U) \right\},$$

that is $\mathit{Diff}_k^s(J)(U)$ is obtained by adding to $J(U)$ the elements $D(f)$ for every $D \in \mathit{Diff}_k^s(U)$ and $f \in J(U)$.

Under this assumptions, it is easy to check that

1. $\mathit{Diff}_k^0(J) = J$, and
2. for any $s \in \mathbb{Z}_{\geqslant 0}$, the following inclusion of sheaves of ideals in $\mathcal{O}_V$ holds,

$$\mathit{Diff}_k^s(J) \subset \mathit{Diff}_k^{s+1}(J).$$

Then, the definition of differential Rees algebras arises in a natural way, only by keeping track of the weight.

Definition 2.12. Given a smooth scheme V, we say that a Rees algebra $\mathcal{G} = \bigoplus_{n\geqslant 0} I_n W^n$, is an *absolute differential algebra* or simply a *differential algebra*, if:

(i) for $n \geqslant 0$, $I_{n+1} \subset I_n$.
(ii) For a suitable affine open covering of V, say $\{U_i\}$, and for every differential operator $D \in \mathit{Diff}^r_k(U_i)$, and $h \in I_n(U_i)$, then

$$D(h) \in I_{n-r}(U_i) \quad \text{if } n \geqslant r.$$

Condition (ii) can be reformulated as:

(ii′) For each n and each $0 \leqslant r \leqslant n$, $\mathit{Diff}^r_k(I_n) \subset I_{n-r}$.

2.13. Local description of differential operators. Let V be a smooth scheme of dimension n over a field k. Given a closed point $x \in V$, consider a regular system of parameters $\{x_1, \ldots, x_n\}$ at $\mathcal{O}_{V,x}$. If k' is the residue field of k (a finite extension of k), the completion is defined by $\widehat{\mathcal{O}}_{V,x} = k'[[x_1, \ldots, x_n]]$.

At the completion level, the Taylor morphism can be defined as the k'-linear continuous ring homomorphism

$$\begin{aligned} \mathit{Tay} : k'[[x_1, \ldots, x_n]] &\longrightarrow k'[[x_1, \ldots, x_n, T_1, \ldots, T_n]] \\ x_i &\longmapsto x_i + T_i \end{aligned}$$

and for any $f \in k'[[x_1, \ldots, x_n]]$,

$$\mathit{Tay}(f) = \sum_{\alpha \in \mathbb{N}^n} g_\alpha T^\alpha, \quad \text{with } g_\alpha \in k'[[x_1, \ldots, x_n]].$$

Here, differential operators appear in the ambit of formal power series: Define, for each $\alpha \in \mathbb{N}^n$, $\Delta^\alpha(f) = g_\alpha$. This Δ^α are defined on the ring of formal power series, but considering the natural inclusion of $\mathcal{O}_{V,x}$ in its completion, it can be proved that $\Delta^\alpha(\mathcal{O}_{V,x}) \subset \mathcal{O}_{V,x}$. Moreover the set

$$\left\{\Delta^\alpha \;\middle|\; \alpha \in \mathbb{N}^n,\ 0 \leqslant |\alpha| \leqslant r\right\}$$

generates Diff^r_k locally at x (for more details of these facts, see Theorem 16.11.2 in [13]).

In the following theorem it is formulated how differential Rees algebras are related to general Rees algebras.

Theorem 2.14. *Let V be a smooth scheme. Given a Rees algebra $\mathcal{G}$ over V, there exists a differential Rees algebra denoted by $G(\mathcal{G})$, such that*:

(i) $\mathcal{G} \subset G(\mathcal{G})$.
(ii) *If $\mathcal{G} \subset \widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}$ is a differential algebra, then $G(\mathcal{G}) \subset \widehat{\mathcal{G}}$.*

Moreover, locally at $x \in V$, a closed point, if $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$, is a local set of generators of $\mathcal{G}$, then

$$\mathcal{F}' = \left\{\Delta^\alpha(f_i) W^{n'_i - \alpha} \;\middle|\; f_i W^{n_i} \in \mathcal{F}, \alpha \in \mathbb{N}^n,\ 0 \leqslant |\alpha| < n'_i \leqslant n_i,\ \text{for } 1 \leqslant i \leqslant s\right\} \tag{1}$$

is a set of generators of $G(\mathcal{G})$ locally at x.

Proof. See [24, Theorem 3.4]. □

Remark 2.15. The previous theorem shows that there exists a smallest differential Rees algebra containing $\mathcal{G}$, it is denoted by $G(\mathcal{G})$.

Remark 2.16. By the local description of $\mathcal{G}$ and $G(\mathcal{G})$, it is easy to check that

$$\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}\big(G(\mathcal{G})\big).$$

2.17. Relative differential algebras. Let $V \xrightarrow{\phi} V'$ be a smooth morphism of smooth schemes. Denote by $\mathit{Diff}^r_\phi(V)$ the locally free sheaf of relative differential operators of order r.

Definition 2.18. A Rees algebra $\mathcal{G} = \bigoplus I_n W^n$ over V is a *ϕ-relative differential algebra*, if

(i) for $n \geqslant 0$, $I_{n+1} \subset I_n$.
(ii) For a suitable affine open covering of V, say $\{U_i\}$, and for every relative differential operator $D \in \mathit{Diff}^r_\phi(U_i)$, and $h \in I_n(U_i)$,

$$D(h) \in I_{n-r}(U_i) \quad \text{if } n \geqslant r.$$

As in Definition 2.18, condition (ii) can be reformulated by:

(ii′) For each n, and $0 \leqslant r \leqslant n$, $\mathit{Diff}^r_\phi(I_n) \subset I_{n-r}$.

Remark 2.19. Since $\mathit{Diff}^r_\phi(V) \subset \mathit{Diff}^r_k(V)$, then any differential algebra is also a ϕ-relative differential algebra.

Remark 2.20. As in Theorem 2.14, any Rees algebra $\mathcal{G}$ can be extended to a smallest ϕ-relative differential algebra. Given an ideal $J \subset \mathcal{O}_V$ and a smooth morphism $V \xrightarrow{\phi} V'$ we can considered the natural extension of ideals, say $J \subset \mathit{Diff}^r_\phi(J)$, defined for each open subset U in V as

$$\mathit{Diff}^r_\phi(J)(U) = \big\{D(f) \bigm| f \in J(U),\ D \in \mathit{Diff}^r_\phi(U)\big\}.$$

Finally, a Rees algebra $\mathcal{G} = \bigoplus I_k W^k$ is a ϕ-relative differential algebra, if and only if $\mathit{Diff}^r_\phi(I_n) \subset I_{n-r}$, for any positive integers $r \leqslant n$.

3. Elimination algebras

In this section we briefly sketch the definition of elimination algebras introduced by Villamayor in [23]. We start given the definition in the universal context together with its specialization. Finally we introduce the definition in the more general case. For further explanations and more detailed expositions, see [23,25,7].

3.1. Universal elimination algebra. Let S be a ring. Consider the polynomial ring $S[Z]$ and a monic polynomial $f(Z) \in S[Z]$ such that $f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n$. As in [23], elimination algebras arise quite naturally when we search for equations on the coefficients which are invariant by all change of variables, namely those of the form $Z \mapsto uZ + s$, where $\alpha \in S$ and $u \in U(S)$. For this reason we discuss some aspects of invariant and elimination theory, and obtain some results in a universal way.

Let define $F_n(Z) = (Z - Y_1)(Z - Y_2)\dots(Z - Y_n)$ as the *universal monic polynomial of degree n* in the polynomial ring of $n+1$ variables $k[Y_1, \dots, Y_n, Z]$. The group of permutations of n elements, $\mathbb{S}_n$, acts on $k[Y_1, \dots, Y_n]$ by permuting the index of the variables $Y_1, \dots, Y_n$; and this action extends to $k[Y_1, \dots, Y_n, Z]$ by fixing Z.

The subring of invariants, say $k[Y_1, \dots, Y_n]^{\mathbb{S}_n}$, is generated, as a k-algebra, by the symmetric elemental functions of order i, say $s_{n,i}$, for $1 \leqslant i \leqslant n$:

$$\begin{aligned} s_{n,1} &= Y_1 + \dots + Y_n \\ s_{n,2} &= \sum_{1 \leqslant i < j \leqslant n} Y_i Y_j \\ &\vdots \\ s_{n,n} &= Y_1 Y_2 \dots Y_n. \end{aligned}$$

That is, $k[Y_1, \dots, Y_n]^{\mathbb{S}_n} = k[s_{n,1}, \dots, s_{n,n}]$ and

$$k[Y_1, \dots, Y_n, Z]^{\mathbb{S}_n} = k[s_{n,1}, \dots, s_{n,n}][Z].$$

It is easy to check that the monic polynomial

$$F_n(Z) = (Z - Y_1) \dots (Z - Y_n) \in k[s_{n,1}, \dots, s_{n,n}][Z],$$

since this polynomial is invariant under the action of $\mathbb{S}_n$.

Let S be a k-algebra and fix $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_n$, a monic polynomial of degree n in $S[Z]$. This polynomial arises from the universal polynomial $F_n(Z)$ by the base change morphism

$$\begin{aligned} k[s_{n,1}, \dots, s_{n,n}] &\longrightarrow S \\ s_{n,i} &\longmapsto (-1)^i \cdot a_i \end{aligned}$$

which induces a morphism

$$k[s_{n,1}, \dots, s_{n,n}][Z] \longrightarrow S[Z]$$

that maps $F_n(Z)$ to $f(Z)$.

Results in the universal setting will provide results for the fixed polynomial $f(Z)$.

The group $\mathbb{S}_n$ acts linearly over the polynomial ring $k[Y_1, \dots, Y_n, Z]$ and this action preserves the grading of the ring. So the invariant subring given by $k[s_{n,1}, \dots, s_{n,n}][Z]$ can be considered as a graded sub-ring (with the grading inherited from that of $k[Y_1, \dots, Y_n, Z]$). The group $\mathbb{S}_n$ also acts linearly in the graded sub-ring $k[Y_i - Y_j]_{1 \leqslant i,j, \leqslant n} \subset k[Y_1, \dots, Y_n]$ defining an inclusion of graded sub-rings

$$k[Y_i - Y_j]_{1 \leqslant i,j, \leqslant n}^{\mathbb{S}_n} \subset k[Y_1, \dots, Y_n]^{\mathbb{S}_n}.$$

Set $k[Y_i - Y_j]^{\mathbb{S}_n} = k[H_{n_1}, \dots, H_{n_r}]$, where the H_{n_i} are homogeneous polynomials of degree n_i, $1 \leqslant i \leqslant r$. Note that H_i is also weighted homogeneous of degree n_i in $k[s_{n,1}, \dots, s_{n,n}]$, where each s_i has degree i, that is,

$$H_{n_i} = H_{n_i}(s_{n,1}, \dots, s_{n,n}).$$

The graded algebra $k[Y_i - Y_j]^{\mathbb{S}_n}$ will be called the *universal elimination algebra*, any polynomial in this ring provides, for each $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_n$ in $S[Z]$, and each base change as above, a function on the coefficients a_i which is invariant by changes of variable of the form $Z \mapsto Z - s$, $s \in S$. In other words, we have obtain functions in the coefficients

$$h_{n_i}(a_1, \ldots, a_n)$$

which are invariants by changes of the form $Z \mapsto Z - s$.

Note that if we consider now more general changes of variables, namely these of the form $Z \mapsto uZ - s$, $s \in S$ and $u \in U(S)$, we obtain that the previous functions are of the form

$$u^{n_i} h_{n_i}(a_1, \ldots, a_n).$$

Nevertheless, we are considering graded algebras, and in particular, for every weight $n \in \mathbb{Z}_{\geqslant 0}$ we will consider an ideal J_n generated by weighted homogeneous polynomials h of degree n and coefficients in S, say $H_n(V_1, \ldots, V_n)$, where each variable V_j has weight n_j and $h = H_n(h_{n_1}, \ldots, h_{n_r})$. So, every ideal J_n of the elimination algebra is invariant by changes of the form $Z \mapsto uZ + s$ even if the functions h_{n_i} are not invariant. It is enough to consider the particular case of changes of the form $Z \mapsto Z - s$.

3.2. The Taylor morphism in the universal setting. A morphism *Tay* is defined as in 2.13. Let S be a k-algebra and consider the S-homomorphism

$$\begin{aligned} Tay : S[Z] &\longrightarrow S[Z, T] \\ Z &\longmapsto Z + T. \end{aligned}$$

For $f(Z) \in S[Z]$, we have

$$Tay\big(f(Z)\big) = \sum_{\alpha \in \mathbb{N}} g_\alpha(Z) T^\alpha,$$

with $g_\alpha(Z) \in S[Z]$ and finally define, for each $\alpha \in \mathbb{N}$, $\Delta^{(\alpha)}(f(Z)) = g_\alpha(Z)$.

Remark 3.3. Since $F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \ldots (Z - Y_n) \in k[Y_1, \ldots, Y_n][Z]$, then

$$F_n(T + Z) = \big(T + (Z - Y_1)\big) \cdot \big(T + (Z - Y_2)\big) \ldots \big(T + (Z - Y_n)\big).$$

The coefficients of this polynomial in the variable T are the symmetric polynomials evaluated on the element $(Z - Y_1, \ldots, Z - Y_n)$. So

$$\Delta^{(\alpha)}\big(F_n(Z)\big) = (-1)^{n-\alpha} s_{n,n-\alpha}(Z - Y_1, Z - Y_2, \ldots, Z - Y_n). \tag{2}$$

Here $\mathbb{S}_n$ acts on the graded sub-ring $k[Z - Y_1, \ldots Z - Y_n] (\subset k[Y_1, \ldots, Y_n, Z])$ setting $\sigma(Z) = Z$ for every $\sigma \in \mathbb{S}_n$ and preserving the graded structure. Note that

$$k[Z - Y_1, \ldots, Z - Y_n]^{\mathbb{S}_n} = k\big[F_n(Z), \Delta^{(\alpha)}\big(F_n(Z)\big)\big]_{1 \leqslant \alpha \leqslant n-1},$$

and that each $\Delta^{(\alpha)}(F_n((Z))$ is homogeneous of degree $n - \alpha$.

As $Y_i - Y_j = (Z - Y_j) - (Z - Y_i)$ we deduce, from the inclusion

$$k[Y_i - Y_j]_{1 \leqslant i, j, \leqslant n} \subset k[Z - Y_1, \ldots Z - Y_n],$$

an inclusion of graded sub-rings

$$k[H_{n_1},\dots,H_{n_r}] = k[Y_i - Y_j]^{\mathbb{S}_n} \subset k\big[F_n(Z), \Delta^{(\alpha)}\big(F_n(Z)\big)\big]_{1\leqslant\alpha\leqslant n-1}, \tag{3}$$

so now, each H_{n_i} is weighted homogeneous in $k[F_n(Z), \Delta^{(\alpha)}(F_n(Z))]_{1\leqslant\alpha\leqslant n-1}$.

3.4. Specialization of the elimination algebra. We will now assign, to the monic polynomial $f(Z) = Z^n + a_1Z^{n-1} + \dots + a_n$ of degree n in $S[Z]$, a Rees algebra which is a sub-algebra of the ring $S[Z][W]$ (i.e., finitely generated sub-algebra of $S[Z][W]$ for a dummy variable W).

To be precise, we attach to a graded subring in $k[s_{n,1},\dots,s_{n,n}][Z]$ a subring in $S[Z][W]$, so that whenever H is a weighted homogeneous polynomial of degree $m \in \mathbb{Z}_{\geqslant 0}$ in $k[s_{n,1},\dots,s_{n,n}][Z]$, we assign to it an element of the form hW^m, with $h \in S[Z]$.

Given $f(Z) = Z^n + a_1Z^{n-1} + \dots + a_n$ in $S[Z]$ we define a k-algebra homomorphism on $S[Z][W]$ by setting

$$\begin{aligned} k[s_{n,1},\dots,s_{n,n}][Z] &\longrightarrow S[Z][W] \\ s_{n,i} &\longmapsto (-1)^i \cdot a_i W^i \\ Z &\longmapsto ZW. \end{aligned}$$

Any graded sub-ring in $k[s_{n,1},\dots,s_{n,n}][Z]$ defines now a graded sub-algebra in $S[Z][W]$, and from (3) we obtain

$$S\big[h_{n_i}W^{n_i}\big] \subset S\big[f(Z)W^n, \Delta^{(\alpha)}\big(f(Z)\big)W^{n-\alpha}\big]_{1\leqslant\alpha\leqslant n-1}. \tag{4}$$

Note that $k[Y_i - Y_j]^{\mathbb{S}_n}$ does not involve the variable Z, so that $S[h_{n_i}W^{n_i}] \subset S[W]$. For this reason, the algebra $S[h_{n_i}W^{n_i}]$ is called the *elimination algebra*.

Remark 3.5. The elements $\Delta^{(\alpha)}(f(Z))W^{n-\alpha}$ in (4) are exactly the relatives differential operators applied to $f(Z)$ with weight $n-\alpha$. Consider the Taylor morphism

$$\begin{aligned} Tay : k[s_{n,1},\dots,s_{n,n}][Z] &\longrightarrow k[s_{n,1},\dots,s_{n,n}][Z,T] \\ Z &\longmapsto Z+T \end{aligned}$$

and the base change morphism $k[s_{n,1},\dots,s_{n,n}] \longrightarrow S$ defined as above. Because of the good behavior of differentials with base change and by (3.2), we obtain $\Delta^{(\alpha)}(f(Z))W^{n-\alpha}$ from the $\Delta^{(\alpha)}(F_n(Z))W^{n-\alpha}$.

3.6. Elimination algebra of two polynomials. A universal elimination algebra was defined in 3.1 for one universal polynomial. These ideas have a natural extension to the case of several polynomials. Here we only consider the case of two polynomials, but the arguments are similar for the case of more than two.

Fix two positive integers, $r, s \in \mathbb{N}$ such that $r + s = n$ and consider $F_r(Z) = (Z - Y_1)\dots(Z - Y_r)$ and $F_s(Z) = (Z - Y_{r+1})\dots(Z - Y_n)$. The permutation group $\mathbb{S}_r$ acts on $k[Y_1,\dots,Y_r]$ and $\mathbb{S}_s$ acts on $k[Y_{r+1},\dots,Y_n]$. Define

$$k\big[H'_{m_1},\dots,H'_{m_{r,s}}\big] := k[Y_i - Y_j]^{\mathbb{S}_r\times\mathbb{S}_s}_{1\leqslant i,j\leqslant n}$$

as *the universal elimination algebra for two polynomials*. Since $\mathbb{S}_r \times \mathbb{S}_s \subset \mathbb{S}_n$, there is a natural inclusion

$$k[H_{m_1},\dots,H_{m_n}] := k[Y_i - Y_j]^{\mathbb{S}_n}_{1\leqslant i,j\leqslant n} \subset k\big[H'_{m_1},\dots,H'_{m_{r,s}}\big],$$

which is a finite extension of graded algebras. On the other hand, one can check that

$$\begin{aligned} &k[Z-Y_1,\ldots,Z-Y_n]^{\mathbb{S}_r\times\mathbb{S}_s} \\ &\quad = k\big[F_r(Z),\Delta^{(\alpha)}\big(F_r(Z)\big),F_s(Z),\Delta^{(\ell)}\big(F_s(Z)\big)\big]_{1\leqslant\alpha\leqslant r-1,\,1,\leqslant\ell\leqslant s-1}. \end{aligned}$$

The inclusion of finite groups $\mathbb{S}_r \times \mathbb{S}_s \subset \mathbb{S}_n$ also shows that there is an inclusion of invariant algebras:

$$\begin{aligned} &k\big[F_n(Z),\Delta^{(j)}\big(F_n(Z)\big)\big]_{j=1,\ldots,n-1} \\ &\quad \subset k\big[F_r(Z),\Delta^{(\alpha)}\big(F_r(Z)\big),F_s(Z),\Delta^{(\ell)}\big(F_s(Z)\big)\big]_{1\leqslant\alpha\leqslant r-1,\,1,\leqslant\ell\leqslant s-1}. \end{aligned}$$

which is a finite extension of graded rings.

Note that $\Delta^{(\alpha)}(F_r(Z))$ is homogeneous of degree $r-\alpha$ for $\alpha = 1,\ldots,n-1$, and that $k[Z-Y_1,\ldots,Z-Y_n]^{\mathbb{S}_r\times\mathbb{S}_s}$ is a graded subring in $k[Y_1,\ldots,Y_, Z]$. Moreover, there is a natural inclusion

$$k\big[H'_{m_1},\ldots,H'_{m_{r,s}}\big] \subset k\big[F_r(Z),\Delta^{(\alpha)}\big(F_r(Z)\big),F_s(Z),\Delta^{(\ell)}\big(F_s(Z)\big)\big]_{1\leqslant\alpha\leqslant r-1,\,1,\leqslant\ell\leqslant s-1} \tag{5}$$

that arises from $k[Y_i - Y_j]_{1\leqslant i,j\leqslant n} \subset k[Z-Y_1,\ldots,Z-Y_n]$.

Here $F_r(Z)F_s(Z) = F_n(Z)$. The rings $k[Y_1,\ldots,Y_r]^{\mathbb{S}_r} = k[v_1,\ldots,v_r]$, and $k[Y_{r+1},\ldots,Y_n]^{\mathbb{S}_s} = k[w_1,\ldots,w_s]$ are graded subrings in $k[Y_1,\ldots,Y_n]$, $F_r(Z) \in k[v_1,\ldots,v_r][Z]$, $F_s(Z) \in k[w_1,\ldots,w_r][Z]$, and there is an inclusion

$$k\big[H'_{m_1},\ldots,H'_{m_{r,s}}\big] \subset k[v_1,\ldots,v_r,w_1,\ldots,w_s]$$

arising from $k[Y_i - Y_j] \subset k[Y_1,\ldots,Y_n]$. In particular each H'_j is also a weighted homogeneous polynomial in the *universal coefficients* $\{v_1,\ldots,v_r,w_1,\ldots,w_s\}$.

3.7. Specialization for two polynomials. The previous discussion, for the case of two polynomials extends to the case of several polynomials. These algebras specialize into sub-algebra of the form

$$S[Z]\big[f_i(Z)W^{n_i},\Delta^{(\alpha_i)}\big(f_i(Z)\big)W^{n_i-\alpha_i}\big]_{1\leqslant\alpha_i\leqslant n_i-1,\,1\leqslant i\leqslant r}, \tag{6}$$

in the sense of (3.4), where $f_i(Z)$ are monic polynomials of the form

$$f_i(Z) = Z^{n_i} + a_{1,i}Z^{n_i-1} + \cdots + a_{n_i,i}$$

for $i = 1,\ldots,r$. The same specialization, applied to the universal elimination algebras (free of the variable Z), define algebras, say

$$S\big[h^{(j)}_{n_1,\ldots,n_r}W^{N^{(j)}_{n_1,\ldots,n_r}}\big]_{1\leqslant j\leqslant R_{n_1,\ldots,n_r}} \subset S[W], \tag{7}$$

for suitable positive integers $R_{n_1,\ldots,n_r}$.

An important property of specializations in (3.4) is their compatibility with finite extensions of graded algebras. So, for example, in the case of two polynomial discussed in 3.6, we conclude that if $f_n(Z) \in S[Z]$ is a monic polynomial of degree n which factorizes as a product of monic polynomials, say $f_n(Z) = f_r(Z)f_s(Z)$, then there is a natural (and finite) inclusion of graded rings:

$$S[Z]\big[f_n(Z)W^n, \Delta^{(j)}\big(f_n(Z)\big)W^{n-j}\big]_{1\leqslant j\leqslant n}$$
$$\subset S[Z]\big[f_r(Z)W^r, \Delta^{(\alpha)}\big(f_r(Z)\big)W^{r-\alpha}, f_s(Z), \Delta^{(\ell)}\big(f_s(Z)\big)W^{s-\ell}\big]_{1\leqslant\alpha\leqslant r-1,\,1,\leqslant\ell\leqslant s-1}$$

(as subalgebras of $S[Z][W]$). Similarly, a finite extension of graded sub-algebras of $S[W]$ is defined by the specialization of the corresponding elimination algebras. The same holds for more than two polynomials.

4. Linear space of vertices

4.1. Let $V^{(d)}$ denote a smooth scheme of dimension d, and let $X \subset V^{(d)}$ be a hypersurface such that $X = V(\langle f\rangle)$ locally at an n-fold point $x \in V^{(d)}$. So $n = \max - \operatorname{ord} f$, the maximum order of the hypersurface in a neighborhood of x. We claim that for a sufficiently *generic* projection

$$V^{(d)} \xrightarrow{\beta} V^{(d-1)}$$

the hypersurface X can be express, in étale topology, as $X = V(f(Z))$, where $f(Z) \in \mathcal{O}_{V^{(d-1)},\beta(x)}[Z]$ is a monic polynomial of degree n in a variable Z. This will hold under a suitable geometric condition, that will be expressed on $\mathbb{T}_{V^{(d)},x}$, the tangent space at the point. In fact, we will show that such conditions on f can be achieved whenever the tangent line, at x, of the smooth curve $\beta^{-1}(\beta(x))$, say $\ell \subset \mathbb{T}_{V^{(d)},x}$, and the tangent cone of the hypersurface at the point, say $\mathcal{C}_f \subset \mathbb{T}_{V^{(d)},x}$, are in general position (intersect only at the origin).

Let $\{x_1,\dots,x_d\}$ be a regular system of parameters at $\mathcal{O}_{V^{(d)},x}$. Recall that

$$\mathbb{T}_{V^{(d)},x} = \operatorname{Spec}\big(gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)},x})\big),$$
$$gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)},x}) = k \oplus \mathfrak{M}_x/\mathfrak{M}_x^2 \oplus \mathfrak{M}_x^2/\mathfrak{M}_x^3 \oplus \cdots \oplus \mathfrak{M}_x^r/\mathfrak{M}_x^{r+1} \oplus \cdots \cong k[X_1,\dots,X_d],$$

where X_i denotes the class of x_i in $\mathfrak{M}_x/\mathfrak{M}_x^2$, that is, the initial form. Let us compute now the initial form of f, to do so consider the following exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle f\rangle & \longrightarrow & \mathcal{O}_{V^{(d)},x} & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & 0\\
0 & \longrightarrow & \mathfrak{M}_x^r \cap \langle f\rangle & \longrightarrow & \mathfrak{M}_x^r & \longrightarrow & \overline{\mathfrak{M}}_x^r & \longrightarrow & 0\\
0 & \longrightarrow & [\mathrm{In}(\langle f\rangle)]_r & \longrightarrow & \mathfrak{M}_x^r/\mathfrak{M}_x^{r+1} & \longrightarrow & \overline{\mathfrak{M}}_x^r/\overline{\mathfrak{M}}_x^{r+1} & \longrightarrow & 0
\end{array}$$

where $[\mathrm{In}(\langle f\rangle)]_r$ denotes the ideal of the homogeneous forms of degree r in the homogeneous ideal $\mathrm{In}(\langle f\rangle)$.

Here $\mathfrak{M}_x^r/\mathfrak{M}_x^{r+1} = \overline{\mathfrak{M}}_x^r/\overline{\mathfrak{M}}_x^{r+1}$ for every $r < n$, and the first time that equality fails to hold is at $r = n$; that is, in degree n, where the initial form of f, say $\mathrm{In}(f)$, appears. So $gr_{\mathfrak{M}}(\mathcal{O}_{X,x}) = k[X_1,\dots,X_d]/\langle \mathrm{In}(f)\rangle$, and the tangent cone of X at x is

$$\mathcal{C}_f = \operatorname{Spec}\big(gr_{\mathfrak{M}_x}(\mathcal{O}_{X,x})\big) = \operatorname{Spec}\big(k[X_1,\dots,X_d]/\big\langle \mathrm{In}(f)\big\rangle\big)(\subset \mathbb{T}_{V^{(d)},x}).$$

4.2. Fix now a smooth morphism $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$, defined at a neighborhood of x, and let ℓ denote the smooth curve $\beta^{-1}(\beta(x))$.

As f has multiplicity n at $\mathcal{O}_{V^{(d)},x}$, the class of f, say $\overline{f}$, has order at least n at the local regular ring $\mathcal{O}_{\ell,x}$. Moreover, the order is exactly n if and only if the tangent line of ℓ at x and $\mathcal{C}_f$ intersect only at the origin of $\mathbb{T}_{V^{(d)},x}$.

If we fix a regular system of parameters, say $\{x_1,\dots,x_{d-1}\}$, at $\mathcal{O}_{V^{(d-1)},\beta(x)}$, the ideal defining ℓ is given by $\langle x_1,\dots,x_{d-1}\rangle$, and a new parameter Z can be added so as to define a regular system of parameters at $\mathcal{O}_{V^{(d)},x}$.

Note finally that the geometric condition imposed at the point x can also be expressed by $\Delta^{(n)}(f)(x)\neq 0$, where $\Delta^{(n)}$ is a suitable differential operator of order n, relative to $\beta : V^{(d)}\to V^{(d-1)}$. The advantage of this new reformulation in terms of differential operators is that it shows that if the geometric condition holds, for X and β at x, it also holds at any n-fold point of X in a neighborhood of x.

Consider now the completion of the local rings in the previous projection, that is, $\widehat{\mathcal{O}}_{V^{(d)},x}$ and $\widehat{\mathcal{O}}_{V^{(d-1)},\beta(x)}$, and apply Weierstrass Preparation Theorem, so the polynomial f can be expressed as

$$u\cdot f(x_1,\dots,x_{d-1},Z)=Z^n+a_1Z^{n-1}+\cdots+a_n,$$

where u is a unit, $\{x_1,\dots,x_{d-1}\}$ is a regular system of parameters at $\widehat{\mathcal{O}}_{V^{(d-1)},\beta(x)}$, and after adding the variable Z, $\{x_1,\dots,x_{d-1},Z\}$ is a regular system of parameters at $\widehat{\mathcal{O}}_{V^{(d)},x}$, and $a_i\in\widehat{\mathcal{O}}_{V^{(d-1)},\beta(x)}$.

A similar result holds replacing completion by henselization. In this case, the coefficients a_i are functions in a étale neighborhood of the point $\beta(x)$ in $V^{(d-1)}$.

4.3. The linear space of vertices. Let $V^{(d)}$ be a smooth scheme, X a hypersurface locally described by f, and $\mathcal{C}_f\subset\mathbb{T}_{V^{(d)},x}$ the tangent cone associated to X at x. Given a vector space $\mathbb{V}$, a vector $v\in\mathbb{V}$ defines a translation, say $tr_v(w)=w+v$ for $w\in\mathbb{V}$. There is a largest linear subspace, say $\mathcal{L}_f$, so that $tr_v(\mathcal{C}_f)=\mathcal{C}_f$ for any $v\in\mathcal{L}_f$, this subspace is called the *inear space of vertices*.

An important property of this subspace $\mathcal{L}_f$ is that for any smooth center Y in X, such that $x\in Y$ and X has multiplicity n along Y, the tangent space of Y, say $\mathbb{T}_{Y,x}$, is a subspace of $\mathcal{L}_f$.

There is a characterization of this linear space in algebraic terms. A homogeneous ideal I is said to be *closed by differential operators* if for any homogeneous element $g\in I$ of order n and any differential operator Δ^α of order $|\alpha|\leqslant n-1$, then $\Delta^\alpha(g)\in I$.

Proposition 4.4. *Let $I=\langle f_1,\dots,f_r\rangle$ be a homogeneous ideal, where each f_i is a homogeneous element of order n_i for $i=1,\dots,r$. There exists a smallest extension of I to an ideal closed by differential operators, say $\widetilde{I}$, given by*

$$\widetilde{I}=\big\langle\Delta^{\alpha_1}(f_1),\dots,\Delta^{\alpha_r}(f_r)\big\rangle_{1\leqslant|\alpha_i|\leqslant n_i-1}.$$

Proof. Any homogeneous element of I can be expressed as a homogeneous combination of products of the generators. So it is enough to consider the case $g=f_i\cdot f_j$, where $i,j\in\{1,\dots,r\}$. We claim that $\Delta^\alpha(g)\in\widetilde{I}$ for every α so that $|\alpha|<n_i+n_j$.

Applying the product rule to g, we obtain

$$\Delta^\alpha(f_i\cdot f_j)=\sum_{\alpha_1+\alpha_2=\alpha}\Delta^{\alpha_1}(f_i)\Delta^{\alpha_2}(f_j).$$

Since $\Delta^{\alpha_1}(f_i)=0=\Delta^{\alpha_2}(f_j)$ for $\alpha_1>n_i$, $\alpha_2>n_j$ and $\Delta^{n_i}(f_i)=\Delta^{n_j}(f_j)=1$, we deduce that $\Delta^\alpha(g)$ is a linear combination of elements of $\widetilde{I}$. $\square$

Remark 4.5. Given a regular system of parameters $\{x_1,\dots,x_d\}$ at $\mathcal{O}_{V^{(d)},x}$, any homogeneous ideal closed by differential operators is defined by

- Linear forms,
- Elements of $k[X_1^p,\dots,X_d^p]$,
- $\dots$,
- Elements of $k[X_1^{p^m},\dots,X_d^{p^m}]$ for some positive integer $m\in\mathbb{Z}_{>0}$.

Suppose now that k is a perfect field, then any homogeneous ideal closed by differential operators $\widetilde{I}$ is, after linear change of variables, of the form

$$\widetilde{I} = \langle X_1, \ldots, X_{\tau_0}, X_{\tau_0+1}^p, \ldots, X_{\tau_1}^p, \ldots, X_{\tau_{m-1}+1}^{p^m}, \ldots, X_{\tau_m}^{p^m} \rangle.$$

If we extend $\langle In(f) \rangle$ to the smallest ideal closed by differential operators, say $\widetilde{I}$, then the zero-set of this homogeneous ideal defines the subspace $\mathcal{L}_f$ we have just defined. In these arguments we are assuming that the underlying field k is perfect.

Similar notions can be defined for Rees algebras. Let $\mathcal{G}$ be a Rees algebra over the smooth scheme $V^{(d)}$. A homogeneous ideal is defined by $\mathcal{G}$ at x, say $In_x(\mathcal{G})$, included in $gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)},x})$; namely that homogeneous ideal generated by the class of I_s at the quotient $\mathfrak{M}_x^s/\mathfrak{M}_x^{s+1}$, for all s. Denote this ideal by $\mathbb{I}_{\mathcal{G},x}$. The ideal $\mathbb{I}_{\mathcal{G},x}$ defines a cone, say $\mathcal{C}_{\mathcal{G}}$, at $\mathbb{T}_{V^{(d)},x}$. Recall that there is a largest subspace, say $\mathcal{L}_{\mathcal{G}}$, included and acting by translations on $\mathcal{C}_{\mathcal{G}}$.

One can check that $In_x(G(\mathcal{G}))$ is the smallest (homogeneous) extension of $\mathbb{I}_{\mathcal{G},x} = In_x(\mathcal{G})$, closed by the action of the differential operators Δ^α; that is, with the property that if F is a homogeneous polynomial of degree N in the ideal, and if $|\alpha| \leqslant N-1$, then also $\Delta^\alpha(F)$ is in the ideal. This homogeneous ideal defines the subspaces $\mathcal{L}_{\mathcal{G}}$, included in $\mathcal{C}_{\mathcal{G}}$, with the properties stated before.

Recall that $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(G(\mathcal{G}))$ (see (2.16)). The previous discussion shows how the homogeneous ideal at x attached to $G(\mathcal{G})$, say $In_x(G(\mathcal{G}))$, relates to the one attached to $\mathcal{G}$, say $In_x(\mathcal{G})$: If $\mathcal{C}_{\mathcal{G}}$ is the cone associated with $\mathcal{G}$, then the cone associated to $G(\mathcal{G})$ is the linear subspace $\mathcal{L}_{\mathcal{G}}$.

Definition 4.6 *(Hironaka's τ-invariant).* $\tau_{\mathcal{G},x}$ will denote the minimum number of variables required to express generators of the ideal $\mathbb{I}_{\mathcal{G},x}$. This algebraic definition can be reformulated geometrically: $\tau_{\mathcal{G},x}$ is the codimension of the linear subspace $\mathcal{L}_{\mathcal{G},x}$ in $\mathbb{T}_{V^{(d)},x}$.

5. The τ-invariant and integral closure of Rees algebras

5.1. In this section, we give an easy proof of the compatibility of the τ-invariant and integral equivalence, i.e., given two integrally equivalent Rees algebras, they have the same τ-invariant. This assertion will be needed in the proof of Theorem 6.4. The proof is based on properties of integrally equivalent algebras and the algebraic definition of the τ-invariant. For alternative proofs see, for example, [18].

Theorem 5.2. *If $\mathcal{G}$ and $\mathcal{G}'$ are two Rees algebras over V with the same integral closure (i.e., $\mathcal{G}$ and $\mathcal{G}'$ are integrally equivalent), then for each $x \in \mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(\mathcal{G}')$, there is an equality between their τ-invariants, that is,*

$$\tau_{\mathcal{G},x} = \tau_{\mathcal{G}',x}.$$

Some auxiliary results will be needed to prove the previous theorem:

Lemma 5.3. *Fix a Rees algebra $\mathcal{G} = \bigoplus I_n W^n$ defined locally at x by the set of generators $\{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$, i.e. $\mathcal{G} = \mathcal{O}_{V,x}[f_1 W^{n_1}, \ldots, f_s W^{n_s}]$. Then the initial ideal is*

$$\mathbb{I}_{\mathcal{G},x} = \langle In_{n_1}(f_1), \ldots, In_{n_s}(f_s) \rangle.$$

Proof. Take $h_n W^n \in I_n W^n$. There exits a weighted homogeneous polynomial of degree n, say $G_n(Y_1, \ldots, Y_s) \in \mathcal{O}_V[Y_1, \ldots, Y_n]$, where each Y_i has weight n_i, such that $G_n(f_1 W^{n_1}, \ldots, f_s W^{n_s}) = h_n W^n$. The initial form of h_n can be expressed in terms of $In_{n_i}(f_i)$, and hence the set $\{In_{n_i}(f_i)\}$ generates every initial form of $In_n(I_n)$ for any n, therefore the equality holds. $\square$

Lemma 5.4. *Considered the Rees algebra defined locally at x as*

$$\mathcal{G} = \mathcal{O}_{V,x}[f_1 W^{n_1}, \ldots, f_s W^{n_s}].$$

Let $N > 0$ be an integer such that N is a common multiple of every n_i, $i = 1, \dots, s$. Denote by $\mathcal{G}_N = \mathcal{O}_V[I_N W^N]$ the Rees ring attached to I_N. Then,

1. $\mathbb{I}_{\mathcal{G}_N,x} = \langle In_N(I_N)\rangle$.
2. $\sqrt{\mathbb{I}_{\mathcal{G}_N,x}} = \sqrt{\langle In_N(I_N)\rangle} = \sqrt{\mathbb{I}_{\mathcal{G},x}}$, *where* $\sqrt{\ }$ *denotes the radical of a given ideal.*

Proof.

1. $\mathbb{I}_{\mathcal{G}_N,x} = \langle In_{mN}(I_{mN})\rangle_{m\geqslant 0}$, and note that
$$\langle In_{mN}(I_{mN})\rangle = \langle In_N(I_N)\rangle^m \subset \langle In_N(I_N)\rangle.$$
2. Let us check $\sqrt{\langle In_N(I_N)\rangle} = \sqrt{\mathbb{I}_{\mathcal{G},x}}$. The left term inclusion is immediate since $\langle In_N(I_N)\rangle \subset \mathbb{I}_{\mathcal{G},x}$. By Lemma 5.3, it is enough to check the other inclusion for the generators of $\mathcal{G}$. As $f_{n_i}^{\alpha_i} \in I_N$ for $\alpha_i = \frac{N}{n_i}$, then $In_{n_i}(f_{n_i})^{\alpha_i} \in In_N(I_N)$ and the equality holds. $\square$

Lemma 5.5. *Set N and $\mathcal{G}_N$ as in Lemma* 5.4. *Then, $\tau_{\mathcal{G},x} = \tau_{\mathcal{G}_N,x}$.*

Proof. By definition, $\tau_{\mathcal{G},x} = \operatorname{codim}(\mathcal{L}_{\mathcal{G},x})$. $\mathcal{L}_{\mathcal{G},x}$ is the linear space of vertices of the zeroset of $\mathbb{I}_{\mathcal{G},x}$, so it is enough to consider the zeroset of $\sqrt{\mathbb{I}_{\mathcal{G},x}}$ to define $\mathcal{L}_{\mathcal{G},x}$. By Lemma 5.4, $\sqrt{\mathbb{I}_{\mathcal{G},x}} = \sqrt{\mathbb{I}_{\mathcal{G}_N,x}}$ so $V(\sqrt{\mathbb{I}_{\mathcal{G},x}})$ and $V(\sqrt{\mathbb{I}_{\mathcal{G}_N,x}})$ have the same subspace of vertices, that is, $\mathcal{L}_{\mathcal{G},x} = \mathcal{L}_{\mathcal{G}_N,x}$. So, finally, $\tau_{\mathcal{G},x} = \tau_{\mathcal{G}_N,x}$. $\square$

Proof of Theorem 5.2. Assume, locally at x, that the Rees algebras are defined by $\mathcal{G} = \bigoplus I_n W^n = \mathcal{O}_V[f_1 W^{n_1}, \dots, f_r W^{n_r}]$ and $\mathcal{G}' = \bigoplus I'_n W^n = \mathcal{O}_V[g_1 W^{m_1}, \dots, g_s W^{m_s}]$. Choose a positive integer N, such that N is a common multiple of every n_i and every m_j, for $i = 1, \dots, r$ and $j = 1, \dots, s$. Consider the Rees rings attached to $\mathcal{G}$ and $\mathcal{G}'$, and defined by $\mathcal{G}_N = \mathcal{O}_V[I_N W^N]$ and $\mathcal{G}'_N = \mathcal{O}_V[I'_N W^N]$, respectively.

It suffices to consider the case where $\mathcal{G} \subset \mathcal{G}'$, so then $I_N \subset I'_N$ and this inclusion is an integral extension of ideals. It follows that $\langle In_N(I_N)\rangle \subset \langle In_N(I'_N)\rangle$ is an integral extension of ideals, as one can check that the conditions of integral dependence hold for the generators. Then, $\sqrt{\langle In_N(I_N)\rangle} = \sqrt{\langle In_N(I'_N)\rangle}$ and by Lemmas 5.4 and 5.5 we conclude that $\tau_{\mathcal{G},x} = \tau_{\mathcal{G}_N,x} = \tau_{\mathcal{G}'_N,x} = \tau_{\mathcal{G}',x}$, which proves the proposition. $\square$

6. The τ-invariant and the elimination algebra

6.1. In this section we present a result relating the τ-invariant of a differential Rees algebra $\mathcal{G}$ with the τ-invariant of the elimination algebra attached to $\mathcal{G}$. In order to prove this result, we show before a local presentation of $\mathcal{G}$ in terms of the elimination algebra. Although this local presentation is very important by itself and has other applications (see also [3,5–7]), we use it here to study the relationship between both τ-invariants.

A weaker relation between the τ-invariants of a differential Rees algebra and its elimination algebra appears in [23, Proposition 5.12].

6.2. Recall that given a Rees algebra $\mathcal{G}$ and a closed point $x \in \operatorname{Sing}(\mathcal{G})$, then a tangent cone, say $\mathcal{C}_{\mathcal{G}}$, at $\mathbb{T}_{V^{(d)},x} = \operatorname{Spec}(gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)},x}))$ is defined by a homogeneous ideal $\mathbb{I}_{\mathcal{G},x}$ in $gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)},x})$ (see 4.3). If $\mathcal{G} = \mathcal{O}_V[f_1 W^{n_1}, \dots, f_s W^{n_s}]$, locally at x, then $\mathbb{I}_{\mathcal{G},x} = \langle In_{n_1}(f_1), \dots, In_{n_s}(f_s)\rangle$ (see Lemma 5.3). It was indicated in 4.3 that there is a largest subspace, say $\mathcal{L}_{\mathcal{G}}$, included and acting by translations on $\mathcal{C}_{\mathcal{G}}$, and $\tau_{\mathcal{G},x}$ is defined as the codimension of this linear subspace. Furthermore, $\tau_{\mathcal{G},x} \geqslant 1$ whenever $\mathbb{I}_{\mathcal{G},x}$ is non-zero. A projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ is said to be transversal to $\mathcal{G}$ at x if the tangent line of the fiber $\beta^{-1}(\beta(x))$ at x is not included in the subspace $\mathcal{L}_{\mathcal{G}}$.

Fix a regular system of parameters $\{y_1, y_2, \dots, y_{d-1}\}$ at $\mathcal{O}_{V^{(d-1)},\beta(x)}$, and choose an element Z so that $\{y_1, y_2, \dots, y_{d-1}, Z\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d)},x}$. Recall that Rees algebras are to be considered up to integral closure. So if the condition of transversality holds one can modify the local generators $\{f_1W^{n_1}, \dots, f_sW^{n_s}\}$ of $\mathcal{G}$ so that each

$$f_i = Z^{n_i} + a_1^{(i)} Z^{n_i-1} + \dots + a_{n_i}^{(i)} \in \mathcal{O}_{V^{(d-1)},\beta(x)}[Z] \tag{8}$$

is a monic polynomial in Z.

Assume now that $\mathcal{G}$ is a differential Rees algebra relative to $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$. In such case $\mathcal{G}$ can be identified, locally at x, by

$$\mathcal{O}_{V^{(d-1)},x}[Z]\big[f_{n_i}(Z)W^{n_i}, \Delta^{(\alpha_i)}\big(f_{n_i}(Z)\big)W^{n_i-\alpha_i}\big]_{1\leqslant\alpha_i\leqslant n_i-1,\, 1\leqslant i\leqslant s},$$

via the natural inclusion $\mathcal{O}_{V^{(d-1)},x}[Z] \subset \mathcal{O}_{V^{(d)},x}$, which is a specialization of a the universal elimination algebra defined for s monic polynomials as in (6). In particular an elimination algebra, say $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d-1)},\beta(x)}[W]$ is defined as in (7).

The following proposition shows a local presentation of a relative differential Rees algebra in terms of a monic polynomial and the elimination algebra. See also [3].

Proposition 6.3 *(Local relative presentation). Let $x \in \operatorname{Sing}(\mathcal{G})$ be a closed point such that $\tau_{\mathcal{G},x} \geqslant 1$. Consider a projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ which is transversal at x. Assume that $\mathcal{G}$ is a β-relative differential algebra and that there exists an element $f_nW^n \in \mathcal{G}$ such that f_n has order exactly n at the local regular ring $\mathcal{O}_{V^{(d)},x}$. Moreover, it can be assumed that $f_n = f_n(Z)$ is a monic polynomial of degree n in $\mathcal{O}_{V^{(d-1)},\beta(x)}[Z]$. Then, at a suitable neighborhood of x:*

$$\mathcal{G} \sim \mathcal{O}_{V^{(d)}}\big[f_n(Z)W^n, \Delta^{(\alpha)}\big(f_n(Z)\big)W^{n-\alpha}\big]_{1\leqslant\alpha\leqslant n-1} \odot \mathcal{R}_{\mathcal{G},\beta},$$

where $\mathcal{R}_{\mathcal{G},\beta}$ in $\mathcal{O}_{V^{(d)}}[W]$ is identified with $\beta^*(\mathcal{R}_{\mathcal{G},\beta})$ and the equivalence $\sim$ is that in Definition 2.9.

Proof. We may assume that $f_n(Z) \in \{f_1W^{n_1}, \dots, f_sW^{n_s}\}$ as in (8). Let us check these assertions in the universal case. In order to simplify notation we consider here the case of two generators (i.e., the case $s = 2$). So consider variables Z, Y_i y V_j over a field k, and

$$F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \dots (Z - Y_n).$$

This polynomial is universal of degree n, and $f_n = f_n(Z)$ is a pull-back of $F_n(Z)$. Let

$$G_m(Z) = (Z - V_1) \cdot (Z - V_2) \dots (Z - V_m)$$

be the universal polynomial of degree m. A natural inclusion $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{G}$ arises from (5).

The permutation group $\mathbb{S}_n \times \mathbb{S}_m$ acts on $k[Z, Y_1, \dots, Y_n, V_1, \dots, V_m]$. This group also acts on

$$S = k[Z - Y_1, Z - Y_2, \dots, Z - Y_n, Z - V_1, Z - V_2, \dots, Z - V_m].$$

The subring of invariants of S, $S^{\mathbb{S}_n \times \mathbb{S}_m}$, is

$$k\big[\Delta^{(\alpha)}\big(F_n(Z)\big), \Delta^{(\alpha')}\big(G_m(Z)\big)\big]_{0\leqslant\alpha\leqslant n-1,\, 0\leqslant\alpha'\leqslant m-1},$$

where $\Delta^{(\alpha)}(F_n(Z))$ is a homogeneous polynomial of degree $n-\alpha$ and $\Delta^{(\alpha')}(G_m(Z))$ is homogeneous of degree $m-\alpha'$. We add a dummy variable W that simply will express the degree. Then, the subring of invariants $S^{\mathbb{S}_n\times\mathbb{S}_m}$ is

$$k\bigl[\Delta^{(\alpha)}\bigl(F_n(Z)\bigr)W^{n-\alpha}, \Delta^{(\alpha')}\bigl(G_m(Z)\bigr)W^{m-\alpha'}\bigr]_{0\leqslant\alpha\leqslant n-1,\, 0\leqslant\alpha'\leqslant m-1}.$$

The universal elimination algebra is defined as the invariant ring of $\mathbb{S}_n\times\mathbb{S}_m$ acting on

$$S' = k\bigl[(Y_2-Y_1),\dots,(Y_n-Y_1),(V_1-Y_1),\dots,(V_m-Y_1)\bigr].$$

The key observation to prove the assertion is that S is spanned by two subrings: $k[Z-Y_1,\dots,Z-Y_n]$ and S'. The subring of invariants on the first is $k[\Delta^{(\alpha)}(F_n(Z))W^{n-\alpha}]_{0\leqslant\alpha\leqslant n-1}$ and the one of the second is the universal elimination algebra.

So, both invariant algebras are included in $S^{\mathbb{S}_n\times\mathbb{S}_m}$; and in order to prove the claim it suffices to show that $S^{\mathbb{S}_n\times\mathbb{S}_m}$ is an integral extension of the sub-algebra spanned by the two invariant subalgebras. To prove this last condition note that S is an integral extension of the sub-algebra spanned by the two invariant subalgebras. This proves the claim. $\square$

Theorem 6.4. *Let $V^{(d)}$ be a smooth scheme of dimension d, $\mathcal{G}$ a differential Rees algebra, $x\in\mathrm{Sing}(\mathcal{G})$ a closed point, and suppose that $\tau_{\mathcal{G},x}\geqslant 1$. Fix a generic projection $V^{(d)}\xrightarrow{\beta}V^{(d-1)}$ (see* 4.1*). Then an elimination algebra $\mathcal{R}_{\mathcal{G},\beta}$ is defined locally at $\mathcal{O}_{V^{(d-1)},\beta(x)}$, and its τ-invariant drops by one, that is,*

$$\tau_{\mathcal{R}_{\mathcal{G},\beta},\beta(x)} = \tau_{\mathcal{G},x}-1.$$

Proof. Fix a regular system of parameters $\{x_1,\dots,x_d\}$ of $\mathcal{O}_{V^{(d)},x}$. So the graded ring is given by $k'[X_1,\dots,X_d]$ where X_i denotes the initial form of x_i. Here k' is the residue field of the local ring at the closed point. If we assume that k' is a perfect field, for a suitable choice of $\{x_1,\dots,x_d\}$, then $\mathbb{I}_{\mathcal{G},x}=\langle X_1^{p^{e_1}},\dots,X_r^{p^{e_r}}\rangle$ for certain non-negative integers e_i, $i=1,\dots,r$ (see Remark 4.5).

So, there exist elements $f_iW^{p^{e_i}}\in\mathcal{G}$ for $i=1,\dots,r$, such that

$$In_{p^{e_i}}(f_i)=X_i^{p^{e_i}}.$$

Since β is generic, then $\mathcal{C}_{\mathcal{G}}\cap\ell=\{0\}$, where ℓ denotes the tangent line to the fiber of the projection. It follows that $\mathcal{C}_{f_i}\cap\ell=\{0\}$ for some $i\in\{1,\dots,r\}$. Assume that this condition is achieved by $i=1$.

Weierstrass Preparation Theorem ensures that in an étale neighborhood of x,

$$f_1(x_1)=x_1^{p^{e_1}}+a_1x_1^{p^{e_1}-1}+\dots+a_{p^{e_1}},$$

where $a_i\in\mathcal{O}_{V^{(d-1)},\beta(x)}$ and the order of each a_i is $>i$. Note that we obtain a strict inequality since $In_{p^{e_1}}(f_1)=X_1^{p^{e_1}}$.

Under this hypothesis, $\nu_x(f_1)=n_1$, and hence Theorem 6.3 applies here so that there is a local relative presentation of the form

$$\mathcal{G}\sim\mathcal{O}_{V^{(d)}}\bigl[f_1(x_1)W^{p^{e_1}},\Delta^{(\alpha)}\bigl(f_1(x_1)\bigr)W^{p^{e_1}-\alpha}\bigr]_{1\leqslant\alpha\leqslant p^{e_1}-1}\odot\beta^*(\mathcal{R}_{\mathcal{G},\beta}).$$

Denote by $\widetilde{\mathcal{G}}$ the Rees algebra defined as

$$\widetilde{\mathcal{G}}=\mathcal{O}_{V^{(d)}}\bigl[f_1(x_1)W^{p^{e_1}},\Delta^{(\alpha)}\bigl(f_1(x_1)\bigr)W^{p^{e_1}-\alpha}\bigr]_{1\leqslant\alpha\leqslant p^{e_1}-1}.$$

Since $In_{p^{e_1}}(f(x_1) = X_1^{p^{e_1}}$, then $\tau_{\widetilde{\mathcal{G}},x} = 1$ and the only variable needed to define the generators of $\mathbb{I}_{\widetilde{\mathcal{G}},x}$ is X_1. On the other hand, the algebra $\beta^*(\mathcal{R}_{\mathcal{G},\beta})$ is free of the variable x_1, so X_1 is not needed to define generators of $\mathbb{I}_{\beta^*(\mathcal{R}_{\mathcal{G},\beta})}$.

Finally, since $\mathcal{G} \sim \widetilde{\mathcal{G}} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta})$ applying Theorem 5.2, it follows that $\tau_{\mathcal{G}} = \tau_{\mathcal{G}_1 \odot \mathcal{R}_{\mathcal{G},\beta}}$, and by the previous arguments, $\tau_{\mathcal{G}_1 \odot \mathcal{R}_{\mathcal{G},\beta}} = 1 + \tau_{\mathcal{R}_{\mathcal{G},\beta}}$. Finally, $\tau_{\mathcal{G}} = 1 + \tau_{\mathcal{R}_{\mathcal{G},\beta}}$. $\square$

In the following corollary, we present the relationship between the τ-invariants in a more general setting; whenever $\mathcal{G}$ is only a relative differential algebra (see [23,7] or [3] for more applications of this relative differential algebras in the problem of resolution of singularities).

Corollary 6.5. *Let $\mathcal{G}$ be a Rees algebra. Fix a transversal projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$, and assume that $\mathcal{G}$ is a β-relative differential algebra. Then the τ-invariants of $\mathcal{G}$ and $\mathcal{R}_{\mathcal{G},\beta}$ are related by*

$$\tau_{\mathcal{R}_{\mathcal{G},\beta},\beta(x)} \leqslant \tau_{\mathcal{G},x} - 1,$$

at any closed point $x \in \operatorname{Sing}(\mathcal{G})$.

Proof. Recall that $\mathcal{G} \subset G(\mathcal{G})$, where G denotes the Giraud's operator (see Theorem 2.14). This inclusion guarantees that $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{R}_{G(\mathcal{G}),\beta}$, where $\mathcal{R}_{G(\mathcal{G}),\beta}$ denotes the elimination algebra attached to $G(\mathcal{G})$. Denote by $\mathcal{H} = \mathcal{R}_{G(\mathcal{G}),\beta}$. The previous inclusion together with Theorem 6.4 imply

$$\tau_{\mathcal{R}_{\mathcal{G},\beta},\beta(x)} \leqslant \tau_{\mathcal{H},\beta(x)} = \tau_{G(\mathcal{G}),x} - 1 = \tau_{\mathcal{G},x} - 1,$$

for any closed point $x \in \operatorname{Sing}(\mathcal{G})$. The last equality holds because of the compatibility of the τ-invariant with the differential operators. $\square$

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